Integration

INTEGRATION BY PARTS

Int. by substitution is the chain rule in reverse.

by parts is the product rule in reverse

REMINDER $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

$$\frac{d}{dx}(x\sin(x)) = 1\cdot\sin(x) + x\cdot\cos(x)$$

IDEA What happens if we integrate both sides and rearrange?

$$\underbrace{\int\frac{d}{dx}}_{\text{cancel out}}(x\sin(x))\,dx = \underbrace{\int 1\cdot\sin(x) + x\cdot\cos(x)\,dx}_{\text{sum rule}}$$

$$x\sin(x) \underset{\text{up to } +C}{=} \int 1\cdot\sin(x)\,dx + \int x\cdot\cos(x)\,dx$$

Now let's rearrange to isolate $\int x\cdot\cos(x)\,dx$.

$$\int x\cdot\cos(x)\,dx = x\sin(x) - \int 1\cdot\sin(x)\,dx$$

$$= x\sin(x) - (-\cos(x)) + C$$

We've taken a derivative, and integrated the product of two things, which we've had no known way of doing so far.

Can we make this more systematic?

GENERALIZATION

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x)$$

$$\int \frac{d}{dx}(f(x)g(x))\,dx = \int f'(x)g(x) + g'(x)f(x)\,dx$$

$$\int \frac{d}{dx}(f(x)g(x))\,dx = \int f'(x)+g(x)\,dx + \int g'(x)f(x)\,dx$$

If we rearrange:

$$\boxed{\int \underline{f(x)}\,\underline{g'(x)}\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

Formula for integration by parts (IBP)

What about the example made it useful? Why did it work well? How can we use it effectively?

$$\int \underline{x} \cdot \underline{\cos(x)}\,dx$$

matches well. Notice

① $x \cdot \cos(x)$ is a product, and we can easily integrate $\cos(x)$.

$(f(x) = x,\ g'(x) = \cos(x))$

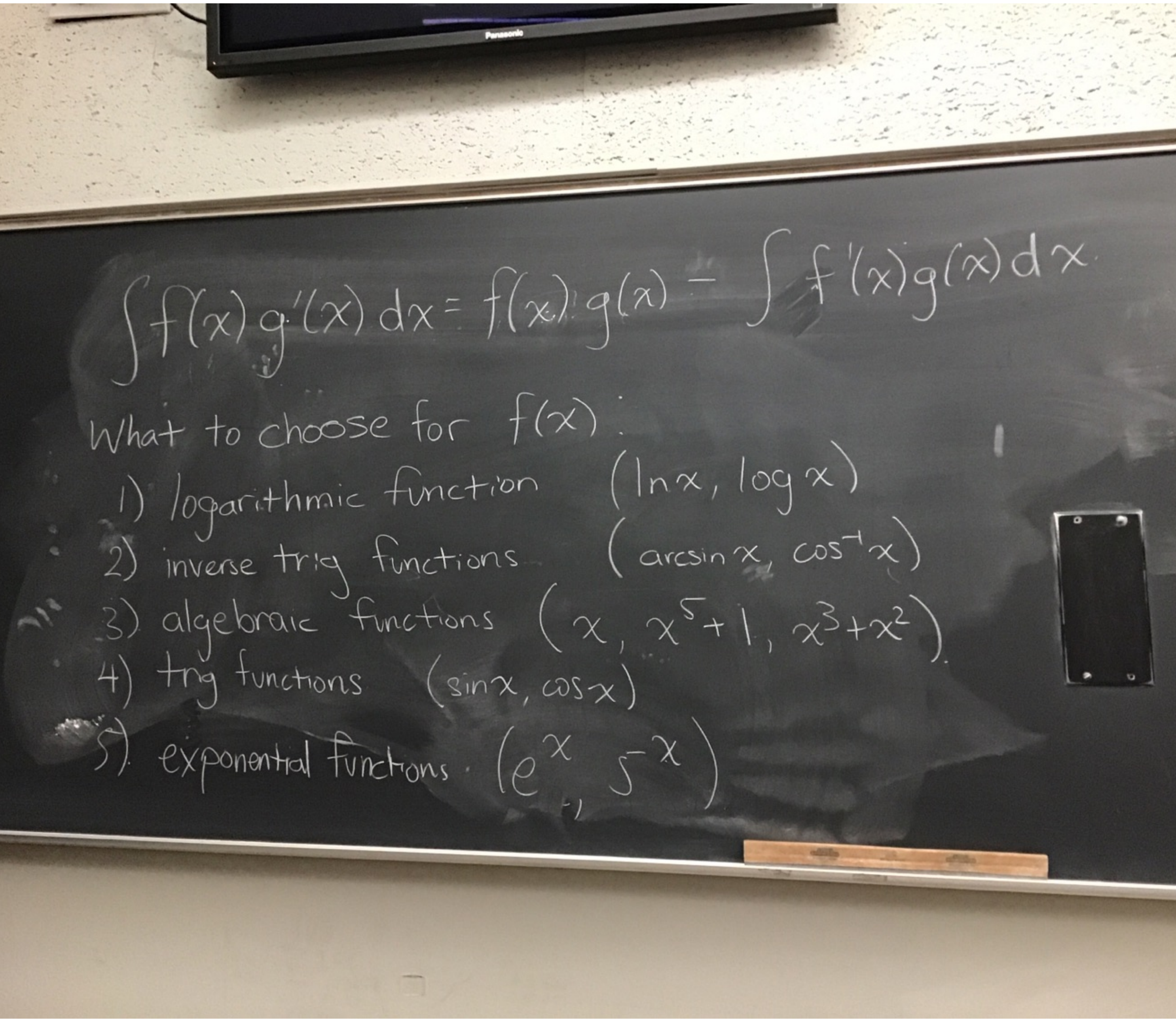
② The product $1 \cdot \sin(x)$ $(= f'(x)g(x))$ is easily integrated.

CONCLUSION

We can use IBP formula if we are integrating a product $f(x)g'(x)$ where

- $g'(x)$ is easily integrated.
- $\int f'(x)g(x)\,dx$ is easier than $\int f(x)g'(x)\,dx$

You need good intuition!



$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx.$$

What to choose for $f(x)$:

1) logarithmic function ($\ln x$, $\log x$)
2) inverse trig functions ($\arcsin x$, $\cos^{-1} x$)
3) algebraic functions (x, x^5+1, x^3+x^2).
4) trig functions ($\sin x$, $\cos x$)
5) exponential functions. (e^x, 5^{-x})

INTEGRATION OF RATIONAL FUNCTIONS

Come up with as many methods as possible to find antiderivatives.

AIM Find a general method to calculate $\int \frac{p(x)}{q(x)}\, dx$

← polynomials

← polynomials

REMINDER $\int \frac{1}{x^2+1}\, dx = \arctan(x) + C$

Let $u = \frac{x}{k}$

$$\Rightarrow \boxed{\int \frac{1}{x^2+k^2}\, dx = \frac{1}{k} \arctan\left(\frac{x}{k}\right) + C}$$

You do not need to prove this

IMPORTANT SPECIAL CASES

① $\int \frac{A}{(ax+b)^j}\, dx$

Let $u = ax+b$

$$\frac{A}{a} \int u^{-j}\, du$$

↑

Power function

② $\int \frac{Bx}{(x^2+k^2)^j}\, dx$

Let $u = x^2+k^2$

(we know the du will cancel the Bx)

$$\frac{B}{2} \int u^{-j}\, du$$

↑

Power function

③ $\int \frac{C}{(x^2+k^2)^j}\, dx$

In prev. lecture, we used I.B.P. to first integrate

$$\int \frac{1}{x^2+1}\, dx \xrightarrow{\text{I.B.P.}} \int \frac{1}{(x^2+1)^2}\, dx \xrightarrow{\text{I.B.P.}} \int \frac{1}{(x^2+1)^3}\, dx \text{ --}$$

||

reduction

we know this is arctan(x)

Very complicated case.

HANDOUT PRACTICE

① $\int \frac{1}{x^2+2x+5}\,dx = \int \frac{1}{(x+1)^2+4}\,dx$

Let $u=(x+1)$ $du = 1\,dx$ $dx = 1\,du$.

$\int \frac{1}{u^2+4}\,du = \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C$

$= \boxed{\frac{1}{2} \cdot \arctan\left(\frac{x+1}{2}\right) + C}$

STRATEGY Reduce all rational integrals to polynomials or these three cases.

There is an important extra class of rational functions that can help us.

PARTIAL FRACTIONS

$\frac{r(x)}{(p(x))^j}$

Properties

① Degree of $r(x)$ is less than degree of $p(x)$ (not $[p(x)]^j$, but p itself)

② $p(x)$ is irreducible (cannot be factored into lower degree polynomials)

Examples of irreducible functions: $2x+1$, x^2+x+1
$(b^2-4ac<0)$

FACT $p(x)$ irreducible ↗ either ↘

linear $p(x) = ax+b$

unbreakable quadratic

$p(x) = ax^2+bx+c$, $b^2-4ac<0$

In the world of polynomials, the equivalent of prime numbers must be either of these cases.

This means that partial fractions must be either of these forms:

$$\frac{A}{(ax+b)^j} \quad \text{OR} \quad \frac{Bx+C}{(ax^2+bx+c)^j} \quad \text{(B could be 0)}$$

where $b^2-4ac<0$

HANDOUT PRACTICE

② Not partial fractions (a) (b)

Partial (c)

(a) $\frac{2x+1}{(3x+2)^3}$ (b) $\frac{3}{x^3-1} = \frac{3}{(x-1)(x^2+x+1)}$

(c) $\frac{6}{(x^2+x+1)^4}$

OBSERVATION

For $\frac{Bx+c}{(ax^2+bx+c)^j}$, if we are smart,

can complete the square to turn it to

$$\frac{Dx+E}{(u^2+k^2)^j} = \underset{\text{(case 2)}}{\frac{Dx}{(u^2+k^2)^j}} + \underset{\text{(case 3)}}{\frac{E}{(u^2+k^2)^j}}$$

CONCLUSION We can always integrate partial fractions

FACT Every rational function can be expressed as a sum of polynomial and partial fractions.

EXAMPLE

$$\frac{x^5+3x^4+7x^3+12x^2+8x+5}{x^4+2x^3+2x^2+2x+1}$$

STEPS

① Do long division. You get

$$(x+1)+\frac{3x^3+8x^2+5x+4}{x^4+2x^3+2x^2+2x+1}$$

② Factor the base into irreducible pieces to get

$$(x+1)+\frac{3x^3+8x^2+5x+4}{\underline{(x+1)^2}\,\underline{(x^2+1)}}$$

The structure of irreducibles tell you the type of partial fraction(s) that appear.

③ Split them up. Polys go up to highest power.

$$(x+1)+\overbrace{\frac{1}{(x+1)}+\frac{2}{(x+1)^2}}+\frac{2x+1}{(x^2+1)}$$

RECALL Overview of PFD and rational integration:

Polynomial

$\deg(R(x)) < \deg(q(x))$

$$\frac{P(x)}{q(x)} = S(x) + \frac{Rx}{q(x)}$$

Factorize into irreducibles

Long division (if deg ↑ > deg ↓)

$$\frac{A}{(ax+b)^j} \qquad \frac{Bx+C}{(ax^2+bx+c)^j}$$

$(b^2 - 4ac < 0)$

(Complete the square + u-sub)

$$\frac{Du}{(u^2+k^2)^j} + \frac{E}{(u^2+k^2)^j}$$

In a test

Not all of this together.

Maybe: a) Just parts of it

b) Describe in words

(Long division will not be asked in a test!)

Example

$$\frac{x^3+2x+1}{(x^2-1)(x+1)^2(x^2+2)^3}$$

① Factor base into irreducibles

$$\frac{x^3+2x+1}{(x-1)(x+1)^3(x^2+2)^3}$$

② Get three partial fractions types:

$$\frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

$$+ \frac{Ex+F}{(x^2+2)} + \frac{Gx+H}{(x^2+2)^2} + \frac{Ix+J}{(x^2+2)^3}$$

③ Find the coefficients by combining the RHS (A ... J part) and set the numerator equal to x^3+2x+1.

(it would be 10 equations and 10 unknowns. will never come out in the exam!)

HANDOUT PRACTICE

① Find partial fraction decomposition for $\frac{2}{1-x^2}$

$$\frac{2}{1-x^2} = \frac{A}{(1+x)} + \frac{B}{(1-x)}$$

$$= \frac{A(1-x) + B(1+x)}{(1+x)(1-x)} = \frac{A - Ax + B + Bx}{(1+x)(1-x)}$$

$$\begin{array}{rl} 2 = & A + B \\ 0 = & -A + B \quad + \\ \hline 2 = & 2A \end{array}$$

$A = 1$, $B = 1$

$$\therefore \frac{1}{(1+x)} + \frac{1}{(1-x)}$$

② Integrate

$\ln|1+x| - \ln|1-x|$

STEP BY STEP (D15)

$\int \frac{p(x)}{q(x)} dx$, where $p(x)$, $q(x)$ are polynomials.

① If degree of $p(x) \geq$ degree of $q(x)$, do long division

② Fully factor the denominator

③ Write the terms of the partial fraction decomposition

factor in denom	terms
$(ax+b)$	$\frac{A}{ax+b}$
$(ax+b)^n$	$\frac{A}{(ax+b)} + \dots + \frac{Z}{(ax+b)^n}$
(ax^2+bx+c)	$\frac{Ax+b}{(ax^2+bx+c)}$
$(ax^2+bx+c)^n$	$\frac{Ax+B}{(ax^2+bx+c)} + \dots + \frac{Yx+Z}{(ax^2+bx+c)^n}$

④ Clear denom

⑤ Match coefficients. Solve.

INTEGRATION OF TRIGONOMETRIC IDENTITIES

AIM calculate $\int \sin^a x \cos^b x \, dx$ where a, b are integers

STRATEGY using simple trig identities, make appropriate substitutions to get rational integrals.

HANDOUT PRACTICE

Why does this work?

③ $\int \sin^{\overset{\text{odd}}{⑤}}(x)\cos^2(x)\, dx$

$$\text{Let } u = \cos(x)$$
$$du = -\sin(x)\,dx$$
$$dx = \frac{du}{-\sin(x)}$$

$$-\int \sin^4(x)\, u \cdot du = -\int (1-u^2)^2 u \cdot du$$
$$= -\int u^2 - 2u^4 + u^6 \, du$$
$$= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C$$

CONCLUSION

① If a odd, choose $u = \cos(x)$

② If b odd, choose $u = \sin(x)$

Works best if odd factor is positive

③ If both even, special case

TRIGONOMETRIC INTEGRALS (Continued)

RECALL $\int \sin^a(x)\cos^b(x)$

① odd $a=2k+1$

$u=\cos(x)$

$-\int (1-u^2)^k u^b\,du$

Best if $a>0$

② $b=2k+1$ odd

$u=\sin(x)$

$\int u^a(1-u^2)^k\,du$

Best if $b>0$

③ Both a, b even

?

MUST $-(a+b)$ even

BEST $-(a+b)>0$

$u=\tan(x)$

Example $\int \sin^2(x)\cos^{-6}(x)\,dx$

DIRTY TRICK Convert to tan and sec $(\tan^c(x)\sec^d(x))$

$$\sin^2(x)\cos^{-6}(x) = \frac{\sin^2(x)}{\cos^6(x)} = \frac{\sin^2(x)}{\cos^2(x)}\cdot\frac{1}{\cos^4(x)} = \tan^2(x)\cdot\sec^4(x)$$

$$\int \tan^2(x)\cdot\sec^4(x)\,dx$$

GENERAL CASE $\sin^a(x)\cos^b(x) = \tan^a(x)\sec^{-(a+b)}X$

Let $u=\tan(x) \Rightarrow u=\frac{1}{\sec^2(x)}dx$

$$= \int \tan^2(x)\overbrace{\sec^4(x)}^{\text{Must be even}}\,dx = \int \tan^2(x)(1+\tan^2(x))\,du$$

$$= \int u^2(1+u^2)\,du = \frac{1}{3}u^3+\frac{1}{5}u^5+C = \boxed{\frac{1}{3}\tan^3(x)+\frac{1}{3}\tan^5(x)+C}$$

SLIGHT ALTERNATIVE TO ③: $\int \cos^2(x)\,dx = \int \sin^0(x)\cos^2(x)\,dx$

$= \int \frac{1}{2}(1+\cos(2x))\,dx$

TRIGONOMETRIC INVERSE SUBSTITUTION

x-world $\longleftrightarrow$ θ-world

$\int f(x)\,dx$

$\longrightarrow \int f(x)\,h'(\theta)\,d\theta$

$= \int f(h(\theta))\,h'(\theta)\,d\theta$

$= F(h(\theta)) + C$

where $\int f = F$

$\underbrace{= F(x) + C}_{\text{Solved in } \theta \text{ world}}$

Usually, $\theta = g(x)$

In inverse,

$x = h(\theta)$

$\frac{dx}{d\theta} = h'(\theta)$

$dx = h'(\theta)\,d\theta$

HANDOUT PRACTICE

① Convert, by inverse substitution, $\int x^3\left(\sqrt{x^2-1}\right)^3 dx$ to a trigonometric integral

$x = \sec(\theta)$

$dx = \sec(\theta)\tan(\theta)\,d\theta$

$\int x^3 \cdot \left(\sqrt{x^2-1}\right)^3 \cdot \sec(\theta)\tan(\theta)\,d\theta$

$= \int x^3 \cdot \left(\sqrt{\sec^2\theta - 1}\right)^3 \cdot \sec(\theta)\tan(\theta)\,d\theta$

$= \int x^3 \cdot \left(\sqrt{\tan^2\theta}\right)^3 \cdot \sec(\theta)\tan(\theta)\,d\theta$

$= \int \sec^3(\theta)\tan^3(\theta) \cdot \sec(\theta)\tan(\theta)\,d\theta$

$= \int \sec^4(\theta)\tan^4(\theta)\,d\theta$

② Integrate it

Let $u = \tan(\theta)$

$du = \sec^2(\theta)\ d\theta$

$d\theta = \frac{du}{\sec^2\theta}$

$\int \sec^4(\theta) \tan^4(\theta) \frac{du}{\sec^2(\theta)}$

$= \int \sec^2(\theta) \cdot \tan^4(\theta)\ du$

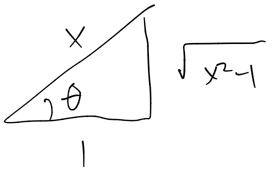
$= \int u^4 (1 + u^2)\ du$

$= \frac{1}{5} u^5 + \frac{1}{7} u^7 + C$

$= \frac{1}{5} (\tan^5(\theta)) + \frac{1}{7} (\tan^7(\theta)) + C$

③ Return to x:

$x = \sec(\theta) \Rightarrow \cos(\theta) = \frac{1}{x}$



$\therefore \tan(\theta) = \sqrt{x^2-1}$

$= \frac{1}{5} (\sqrt{x^2-1})^5 + \frac{1}{7} (\sqrt{x^2-1})^7 + C$

STRATEGIES OF INTEGRATON

Core technique

Reduce every problem to $\overset{x^a}{\underline{\text{power}}}$, $\overset{b^x}{\underline{\text{exponential}}}$ or $\overset{\sin(x),\ \cos(x)}{\underline{\text{trigonometric}}}$ functions using sum rule, constant multiple rule, substitution or integration by parts

Important Special Classes

Rational Integrals $\displaystyle\int \frac{p(x)}{q(x)}\,dx$, where $p(x)$ and $q(x)$ are polynomials

break them into easier-to deal with partial functions

There is a way to turn this into above

Trigonometric Integrals $\displaystyle\int \sin^a(x)\cos^b(x)$

Use substitution, or change into $\displaystyle\int \tan^a(x)\sec^{-a+b}(x)\,dx$

There is a way to turn this into above

Certain Algebraic Integrals $\displaystyle\int x^c\left(\sqrt{r^2 \pm x^2}\right)^d dx$, $\displaystyle\int x^c\left(\sqrt{x^2 - r^2}\right)^d dx$, where c and d are integers, and d is odd

Use inverse trig substitution of special identities

Special Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1 \Rightarrow \underline{r^2 - (r\sin\theta)^2} = (r\cos\theta)^2$$

$$1 + \tan^2(\theta) = \sec^2(\theta) \Rightarrow \underline{r^2 + (r\tan\theta)^2} = (r\sec\theta)^2$$

$\underline{\sqrt{r^2 - x^2}} \Rightarrow$ Let $x = r\sin\theta$

Same form

$\underline{\sqrt{r^2 + x^2}} \Rightarrow$ Let $x = r\tan\theta$

$\underline{\sqrt{x^2 - r^2}} \Rightarrow$ Let $x = r\sec\theta$

$$(r\tan\theta)^2 = \underline{(r\sec\theta)^2 - r^2}$$

FACTS Don't use this as a given fact

$$\int x^c\left(\sqrt{r^2-x^2}\right)^d dx \xrightarrow{x=r\sin(\theta)} \int r^{c+d+1}\sin^c(\theta)\cos^{d+1}(\theta)\,d\theta$$

$$\int x^c\left(\sqrt{r^2+x^2}\right)^d dx \xrightarrow{x=r\tan(\theta)} \int r^{c+d+1}\tan^c(\theta)\sec^{d+2}(\theta)\,d\theta$$

$$\int x^c\left(\sqrt{x^2-r^2}\right)^d dx \xrightarrow{x=r\sec(\theta)} \int r^{c+d+1}\tan^{d+1}(\theta)\sec^{c+1}(\theta)\,d\theta$$

HANDOUT PRACTICE

① a) $\int \frac{1}{\sqrt{1+x^2}}\,dx \Rightarrow$ Let $x = \tan(\theta)$ $\quad dx = \sec^2(\theta)\,d\theta$

$d\theta = \frac{dx}{\sec^2(\theta)}$

$$\int \frac{1}{\sqrt{1+\tan^2(\theta)}}\,dx = \int \frac{1}{\sec\theta}\cdot\sec^2(\theta)\,d\theta$$

$$= \int \sec(\theta)\,d\theta$$

b) $\int \frac{\sec(\theta)\left(\sec(\theta)+\tan(\theta)\right)}{\sec(\theta)+\tan(\theta)}\,d\theta \Rightarrow$ Let $u = \sec(\theta)+\tan(\theta)$

$du = \sec(\theta)\tan(\theta) + \sec^2(\theta)\,d\theta$

$d\theta = \frac{du}{\sec(\theta)\left(\tan(\theta)+\sec(\theta)\right)}$

$$= \int \frac{\cancel{\sec(\theta)}\cdot\cancel{u}}{\cancel{u}}\cdot\frac{du}{\cancel{\sec(\theta)}\;u}$$

$$= \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|\sec(\theta)+\tan(\theta)\right| + C$$

$$\sec(\theta) = \sqrt{\tan^2(\theta)+1}$$

$$= \sqrt{x^2+1}$$

$$\Rightarrow \ln\left|\sqrt{x^2+1} + x\right| + C$$

OR

$$\int \sec(\theta)\,d\theta =$$

$$\int \sin^0(\theta)\cos^{-1}(\theta)\,d\theta$$

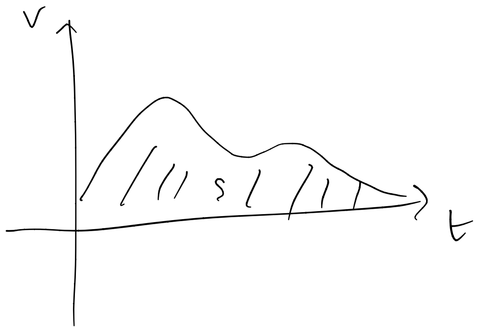
Let $u = \sin(\theta)$

$d\theta = \frac{du}{\cos(\theta)}$

$$= \int \frac{1}{\cos^2(\theta)}\,du = \int \frac{1}{1-u^2}\,du$$

APPROXIMATE INTEGRATION

In certain cases, we may not have a traditional "function" to work with. What if we only have a table of data? e.g.



To find displacement $\int_0^t v(t)\,dt$

Riemann Sum

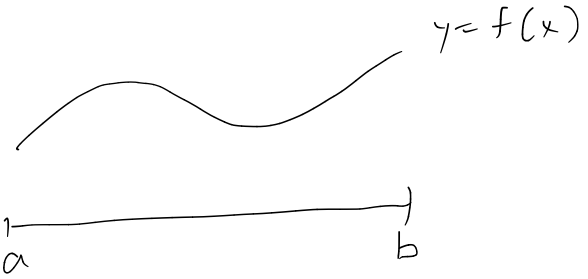
RECALL $\int_a^b f(x)\,dx \Rightarrow$ a limit $\lim_{k\to\infty} S_n$

$$S_n = f(x_1^*)\,\Delta x + f(x_2^*)\,\underbrace{\Delta x}_{\frac{b-a}{n}} + \dots + f(x_n^*)\,\Delta x$$

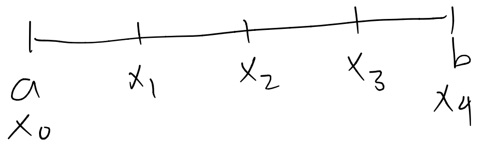
$$= \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

What is Δx?

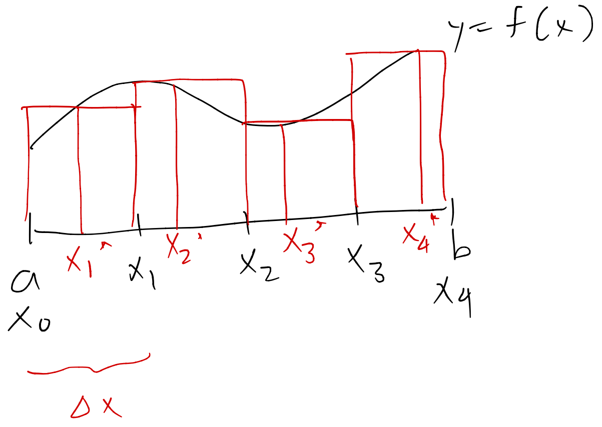
Example $(n=4)$



Divide into 4 equal sub-integrals



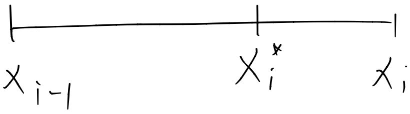
Look at height of each rectangle. Select arbitrary * point for height.



$$S_n = \underbrace{f(x_1^*)\Delta x}_{\text{Area of 1st rectangle}} + f(x\ ^*)\Delta x + \dots + f(x_n\ )\Delta x$$

The bigger the n, the more accurate our prediction.

$\boxed{\text{Example}}$ (Basic)



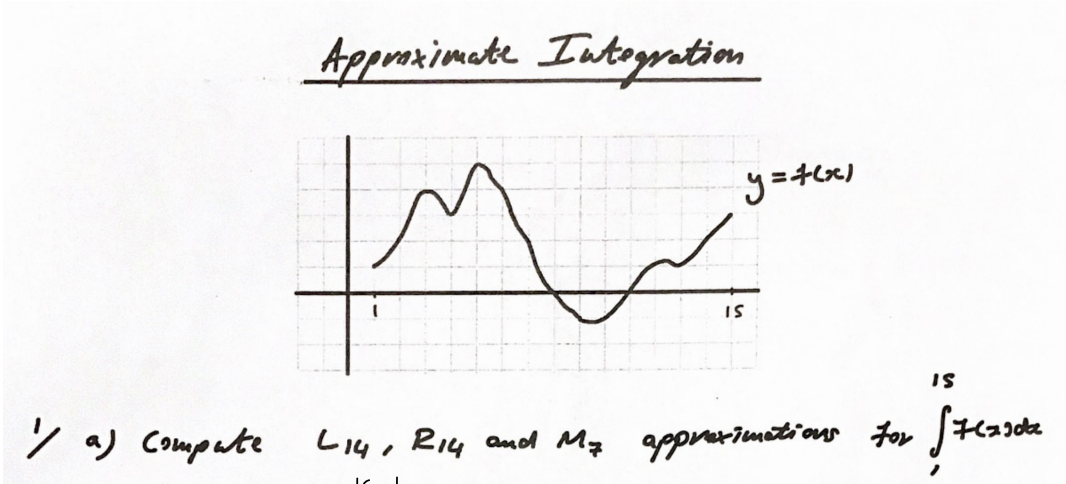
(the i-th subintegral)

Left endpoint : $x_i^* = x_{i-1}$ ← $L_n = S_n$

Right endpoint : $x_i^* = x_i$ ← $R_n = S_n$

Midpoint : where x_i^* is at $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1}+x_i)$ ← $M_n = S_n$

HANDOUT PRACTICE



Approximate Integration

1/ a) Compute L_{14}, R_{14} and M_7 approximations for $\int_1^{15} f(x)dx$

$L_{14} \Rightarrow \Delta x = \frac{15-1}{14} = 1$

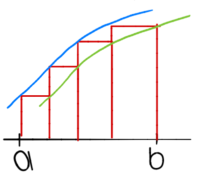
$= 1+2+4+3+5+4+2+0-1-1+0+1+1+2 = 23$

$R_{14} = 2+4+3+5+4+2+0-1-1+0+1+1+2+3 = 25$

$M_7 \Rightarrow \Delta x = 22$

What properties of the graph influence these observations?

OBSERVATION collection of rectangles



under estimate

for strictly increasing f, $L_n < \int_a^b f(x)\,dx$

$R_n > \int_a^b f(x)\,dx$

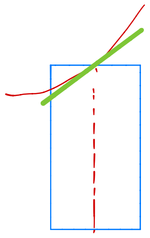
decreasing f, $L_n > \int_a^b f(x)\,dx$

$R_n < \int_a^b f(x)\,dx$

b) Is the R_{100} approximation for $\int_5^9 f(x)dx$ an over or under approximation?

Under.

Midpoint

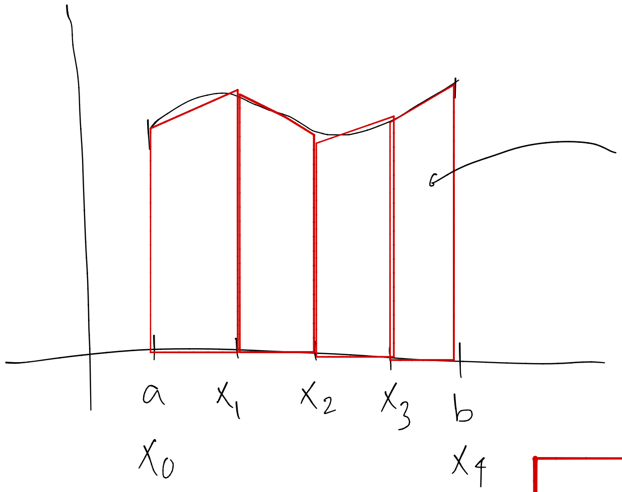


tangent line

Take the line and rotate about the pivot

It becomes shaped as a **trapezoid!**

There are other kinds of approximations: one is **trapezoidal approximation**



$n = 4$

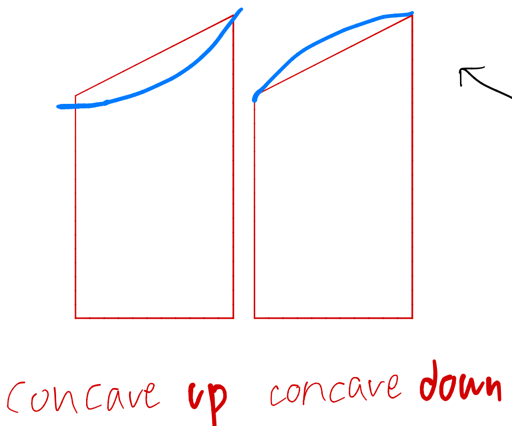
Area $= \frac{1}{2}(f(x_{n+1}) + f(x_n))\ \Delta x$

Thus, $T_n = \frac{1}{2}(f(x_1) + f(x_2))\Delta x + \dots$

$+ \frac{1}{2}(f(x_{n-1}) + f(x_n))\Delta x$

$$\boxed{T_n = \frac{1}{2}\Delta x(f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n))}$$

Is this an over or under approximation?



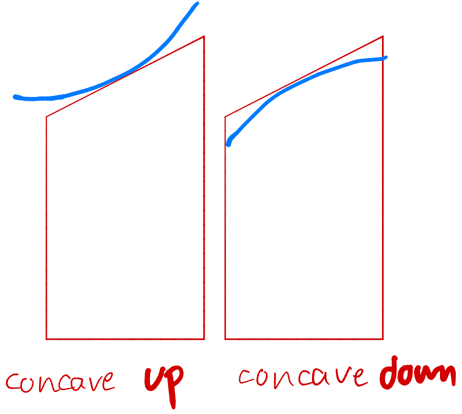
Depends on the curve.

up ⇒ $T_n > \int_a^b f(x)dx$

down ⇒ $T_n < \int_a^b f(x)dx$

2nd derivative tells whether up or down

That means the midpoint approximation is a very specific kind of trapezoidal approximation



concave up ⇒ $M_n < \int_a^b f(x)\,dx$

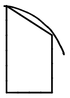
concave down ⇒ $M_n > \int_a^b f(x)\,dx$

HANDOUT PRACTICE

Is M_{2020} an over or underestimate for $\int_2^{10} \cos(x) + x^2\,dx$

$\frac{d}{dx} = -\sin(x) + 2x$

$\frac{d^2}{dx^2} = -\cos(x) + 2$ (+). Going down.



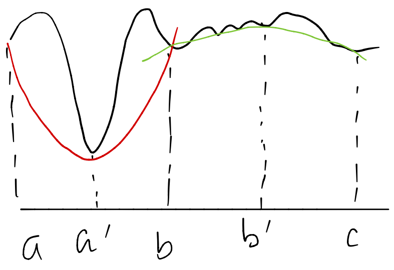
Under estimate

C.U. [2, 10] ⇒ $M_{2020} < \int_2^{10} \cos(x) + x^2\,dx$

(+ at both 2, 10, every point in between)

SIMPSON'S APPROXIMATION

Example



Instead of drawing straight lines, we use the simplest form of curved line – a parabola.

With parabolas, we need 3 distinct points to uniquely determine them.

Formula:

$$S_n = \left(\begin{array}{c}\text{Area under}\\ \text{1st parabola}\end{array}\right) + \dots + \left(\begin{array}{c}\text{Area under}\\ k^{th}\text{ parabola}\end{array}\right)$$

For $n = 2k$

$$S_n = \frac{1}{3}\Delta x\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 4f(x_{n-1}) + 4f(x_n)\right]$$

ERRORS

$$E_{M_n} = \int_a^b f(x)\,dx - M_n$$

nth midpoint error

Every approximation has an error.

Three cases:

$$E_{M_n} = \int_a^b f(x)\,dx - M_n$$

$$E_{T_n} = \int_a^b f(x)\,dx - T_n$$

$$E_{S_n} = \int_a^b f(x)\,dx - S_n$$

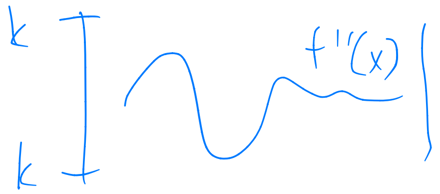
What we want is some control over the size of the error?

DEEP FACT If your graph is a straight line, your error in approx. is zero.

If sth. is straight, its derivative is zero.

The <u>midpoint error bound</u> is thus tightly bounded by the 2nd derivative of a curve.

$|f''(x)| \leq k$



(k encloses f''(x))

$$\Rightarrow |E_{M_n}| \leq \frac{k(b-a)^3}{24n^2}$$

<u>Trapezoidal</u>

$$\Rightarrow |E_{T_n}| \leq \frac{k(b-a)^3}{12n^2}$$

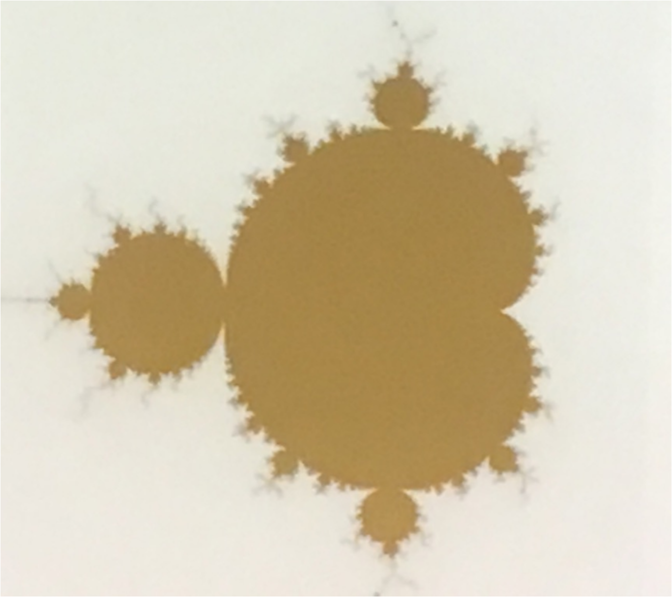
<u>Simpsons</u> — makes no sense to use $f''(x)$. If $f(x)$ is quadratic, we expect 0 error.

$$\Rightarrow |f^4(n)| \leq k \Rightarrow E_{S_n} \leq \frac{k(b-a)^5}{180n^4}$$

IMPROPER INTEGRALS

Let's talk about infinity. Our intuition for infinity often leads us to make incorrect assumptions about concepts related to it.

There exists shapes with a finite area, but an infinite perimeter. How?

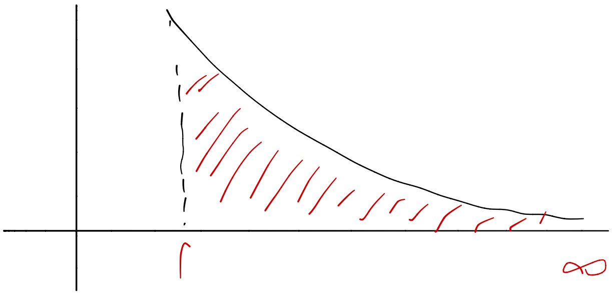


Mandelbrot set

It is a fractal. Most certainly has a finite area, but the edges contribute to an infinite length.

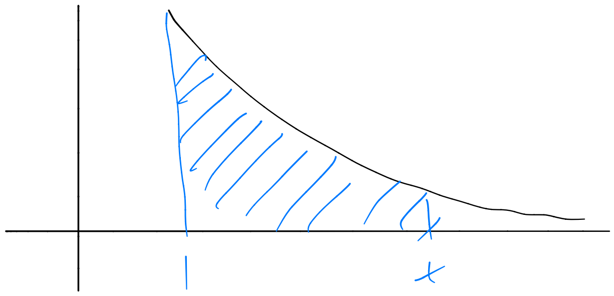
Q Can we find such a shape using calculus?

Example What is the area under $y = \frac{1}{x^2}$ over $[1, \infty]$



Because ∞ is not a definite number, we cannot apply the FTC like w/ definite integrals.

STRATEGY



$\int_1^t \frac{1}{x^2}\,dx \Rightarrow$ As t increases, area becomes closer to $\int_1^\infty \frac{1}{x^2}\,dx$.

Area ▨ $= \lim_{t\to\infty}$ Area ▨

We're bootstrapping up to ∞

$$= \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t$$

$$= \lim_{t\to\infty} 1-\frac{1}{t} = 1 \text{ (as } \tfrac{1}{\infty} \text{ app. } 0)$$

DEFINITION Type 1 Improper Integral

A/ $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$

B/ $\int_{-\infty}^b f(x)\,dx = \lim_{t\to\infty}\int_t^b f(x)\,dx$

Convergent if limit exists. Divergent if not.

C/ $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_b^\infty f(x)\,dx$

is convergent only if are both convergent

HANDOUT PRACTICE

① a Is $\int_1^\infty \frac{1}{x}\,dx$ convergent or divergent?

$$\lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln|t|$$

$\frac{1}{x}$ and $\frac{1}{x^2}$ are very different.

$$\ln|\infty| \Rightarrow \boxed{\text{Divergent}}$$

b $v(x) = \frac{1}{x}$ m/s, $x > 1$ 2nd object. ahead of it, static

will they collide?

No, Divergent.

As long as t is large enough, object can travel over any distance up to ∞.

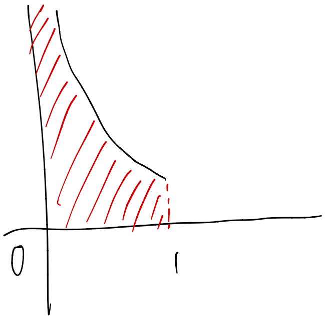
c. Is it the same for $v(x) = \frac{1}{x^2}$?

No. $\frac{1}{x^2}$ is convergent.

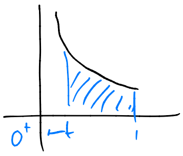
Object cannot travel more than 1m over <u>any</u> interval of time.

$\boxed{\text{Example}}$

What is the area under $y = \frac{1}{\sqrt{x}}$ under $[0, 1]$?

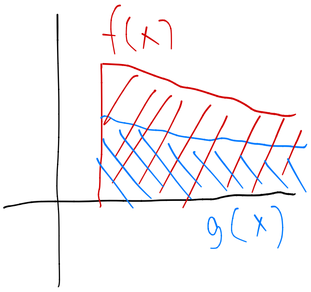


The ∞ is along the vertical asymptote.



RECALL We discussed the comparison theorem, which allows us to get the convergence or divergence of an improper integral, by comparing it to a known, proper integral.

If $0 \leq g(x) \leq f(x)$ on $(0, \infty)$ then



Area (▨) ≤ Area (▨)

A $\int_0^\infty g(x)\,dx$ divergent $\Rightarrow \int_0^\infty f(x)\,dx$ divergent

B $\int_0^\infty f(x)\,dx$ convergent $\Rightarrow \int_0^\infty g(x)\,dx$ convergent

Good to know Improper Integrals

1) $\int_1^\infty \frac{1}{x^p}\,dx =$ conv $p > 1$; div $p \leq 1$

2) $\int_0^1 \frac{1}{x^p}\,dx =$ conv $p < 1$; div $p \geq 1$

3) $\int_0^\infty \frac{1}{b^x}\,dx =$ conv $b > 1$

Example

$$\int_1^\infty \frac{\sin(x) + 2}{x^2}\,dx$$

Prof. Paulin says

Can't integrate $\frac{\sin(x)}{x^2}$. Must do comparison test.

Very vaguely looks like case 1B. Must bound it by something above, by

$0 \le \sin(x)+2 \le 3$

$0 \le \frac{\sin(x)+2}{x^2} \le \frac{3}{x^2}$

$\int_1^\infty \frac{3}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{3}{x^2}dx$

$= \lim_{t\to\infty} -\frac{3}{x}\Big|_1^t = 3-\frac{3}{t} = 3$

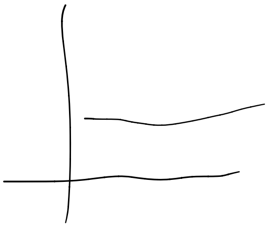
$\int_1^\infty \frac{3}{x^2}dx \ \underline{\text{conv}} \Rightarrow \int_1^\infty \frac{\sin(x)+2}{x^2}dx$

something convergent.

$\int \frac{1}{x^2}$ may fail, as $\sin(x)+x^2$ is generally larger than it. However, $\int \frac{A}{x^2}$ is still convergent.

Check convergence of this.

State this.



HANDOUT PRACTICE

Is $\int_0^1 \frac{\sec(x)}{x}dx$ convergent or divergent?

$0 \le \cos(x) \le 1 \Rightarrow 0 \le 1 \le \sec(x)$

$\Rightarrow 0 \le \frac{1}{x} \le \sec(x)$ $(0,1)$

$\int_0^1 \frac{1}{x}dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}dx = \lim_{t\to 0^+} \ln|x|\Big|_t^1 = \lim_{t\to\infty} -\ln|t| = \infty$

$\int_0^1 \frac{1}{x}dx \ \underline{\text{div}} \Rightarrow \int_0^1 \frac{\sec(x)}{x}dx$ divergent